Pseudocircular Strings

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1 Introduction

Definition: A **string** is a simple bounded arc. We consider collections of strings embedded in $\mathbf{S}^2$. Such a collection is denoted by Σ, which will be extended at each string from their endpoints to **pseudocircles**, simple closed curves.

Definition: A **pseudocircular arrangement** is a extension of Σ to pseudocircles, where every string e is extended to pseudocircle γ_e so that

PS1: For each edge/string e, no vertex except an endpoint of e is contained in γ_e.
PS2: For distinct e, f, $|\gamma_e \cap \gamma_f| = 2$, and all intersections are crossings (no tangent points).
PS3: For any edge e, if its endpoints u and v are contained in the closure of Δ of one of the components of $\mathbf{S}^2 \setminus \gamma_e$, then $e \subset \Delta$.

We can also define PS2': For distinct e, f, $|\gamma_e \cap \gamma_f| \geq 2$, and all intersections are crossings (no tangent points).

Note: In this context, if the two ends of any string are both contained or both not contained in any pseudocircles, then the entire string is contained or not contained in that pseudocircle, respectively. This also prevents any two pairs of strings from intersecting each other more than once, since PS2 limits intersections to two and exactly two intersections would place both ends of the string on one side of the pseudocircular extension of the other string, but the entire string would not be on that side.

Pseudolines are images of lines in planes under homeomorphic functions, and a drawing of a graph is pseudolinear if the edges can be extended to an arrangement of pseudolines.

Definition: For any drawing, we define the **extension arc** of e to be $\gamma_e \setminus e$.

We will answer the following three questions:

1. In any pseudocircular arrangement, are the strings that are contained completely on one side of any pseudocircle pseudolinear?

2. For any arrangement of strings which has a pseudocircular extension which satisfies PS1, PS2', PS3, does it necessarily have another extension which satisfies PS1, PS2, PS3?

3. Can we characterize all arrangements of strings which do not have pseudocircular extensions, similarly to the result from [1] for pseudolinear extensions?

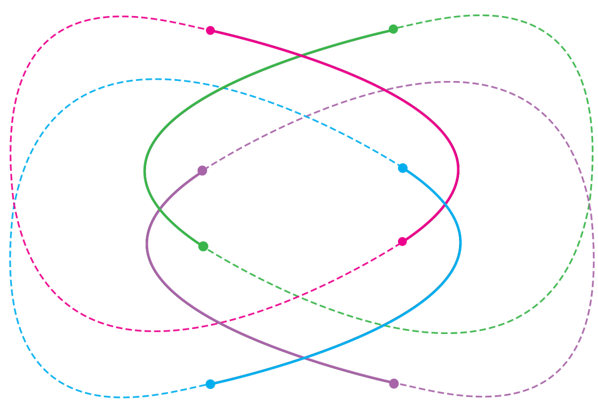
2 Answering Q1

Definition: A **rainbow vertex** of an obstruction cycle is a vertex with all of its incident edges inside the cycle belonging to different strings. [1]

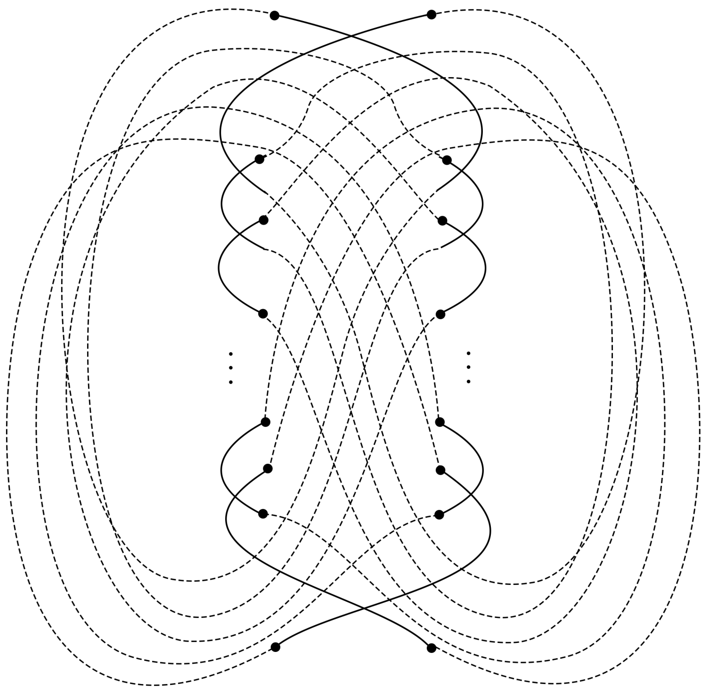
Theorem: [Arroyo, Bensmail, Richter 2018] Arrangments of strings are pseudolinear if and only if they do not contain obstruction cycles, i.e. cycles of strings with at most two rainbow vertices. [1]

Using this result, in order for us to prove that all strings contained entirely within one side of a pseudocircle are pseudolinear, we can equivalently show that there are no obstruction cycles formed by strings on one side of any peudocircle. For these strings to be entirely on one side of a reference pseudocircle, PS3 tells us the reference pseudocircle does not intersect any string; taken with PS2, we find that the reference pseudocircle must intersect each pseudocircle extensions twice on the extension arcs. Consider any pseudolinear obstruction cycle C. For every string on the obstruction cycle, it must have some part of its extension outside of C, that is, on the same side of C as the reference pseudocircle.

The answer to Q1 is no, but only for one class of obstruction. We will discuss this in detail in the next section. The example below gives us a pseudolinear obstruction which can be extended to be contained within a bounding pseudocircle γ:



We can always make a similar construction for cycles with two rainbow non-adjacent vertices; the general case is illustrated in the figure below. It turns out that these will be the only class of pseudolinear obstructions that work; we cannot extend pseudolinear obstruction cycles with adjacent rainbow vertices or at most one rainbow vertex to be contained in a bounding pseudocircle.



For the remainder of this paper, we will refer to pseudolinear obstruction cycles with no rainbow vertices as **clouds**, with one rainbow vertex as **fish**, with two adjacent rainbow vertices as **shrubs**, and with two non-adjacent rainbow vertices as **croissants**.

3 Answering Q2

3.1 The main result

Consider any obstruction cycle C with no rainbow vertices. Assume, for a contradiction, that there is no pseudocircle extension of the strings on C contained entirely within C. Then we can assume the extensions of every string on C exits the cycle from each end initially on another string in the cycle.

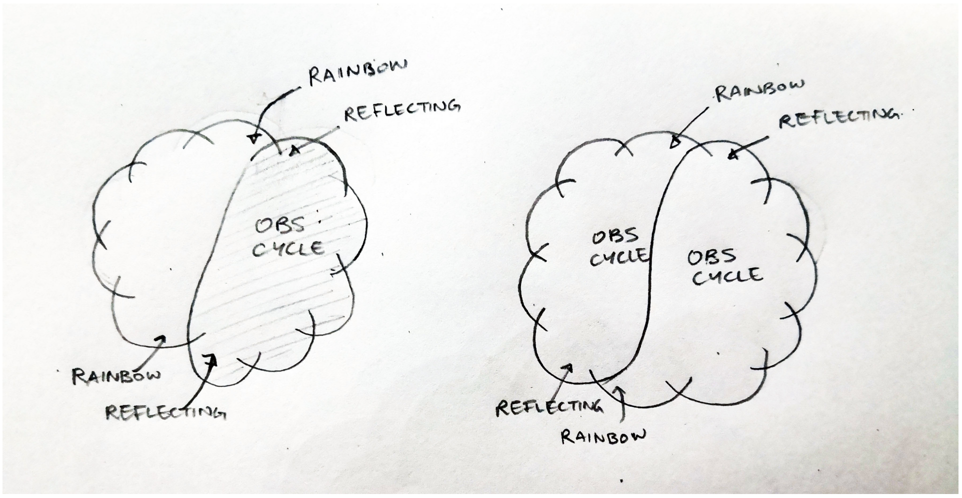
Definition For a given string e, we will refer to the strings on C where the extension arc of e first crosses C, say at x and y, as the **exits** of e, and we will say string e is exiting on string x and y.

Lemma 1. The following are true for any string e on an obstruction cycle C in a pseudocircular set of strings:
i) γ_e does not exit on its neighbours, i.e. strings e is adjacent to on C.
ii) If γ_e is not contained in C, then its exits are distinct.

Proof: In both cases, note that either both endpoints of a neighbour of e or both endpoints of the string e exits on will be on the same side of γ_e which violates PS3, unless γ_e crosses the string, which gives us at least 3 crosses and violates PS2.

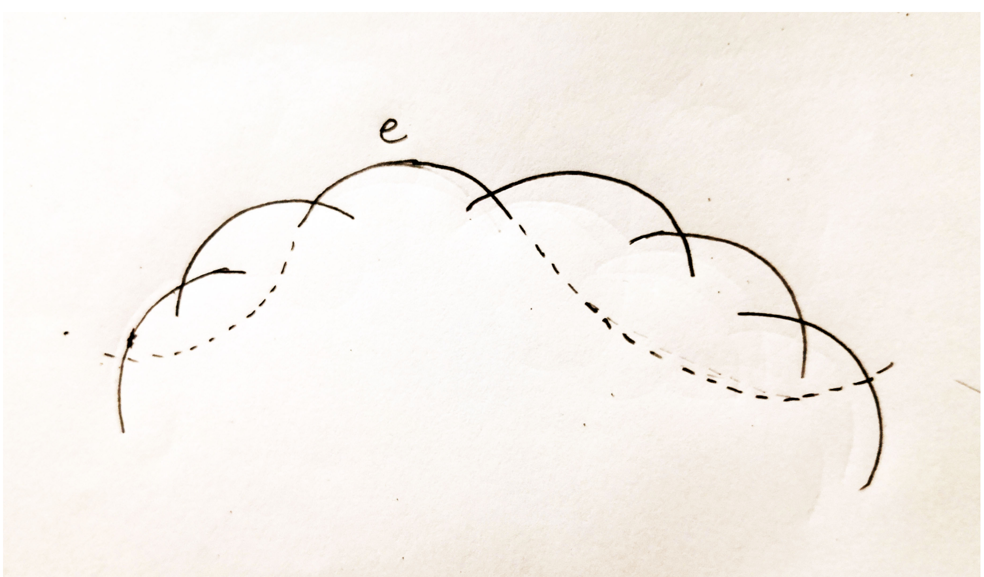
Proposition 2. A cloud, fish, or shrub obstruction cycle entirely contains the pseudocircle extension of one of its strings.

Proof: We assume that all strings going along C in order are distinct. If not, then we can always reduce it to another obstruction cycle contained within the first, as shown below.

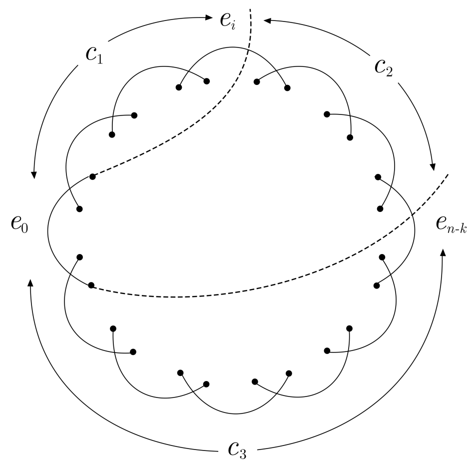


If we break C into two smaller cycles on either side of the edge occurring twice, we get two new rainbow vertices. If the original cycle has at most two vertices, now at least one of the smaller cycles has at most two rainbow vertices.

Assume, for a contradiction, that each string along C does not have its pseudocircular extension contained entirely within C. Then each string must have two exits somewhere on C. For each string e, we define the *exit distances* with respect to γ_e to be the distances along the cycle to the two exits of e. For example, in the figure below, γ_e has exit distances 2 and 3. Lemma 1 implies that exit distances are at least 2.

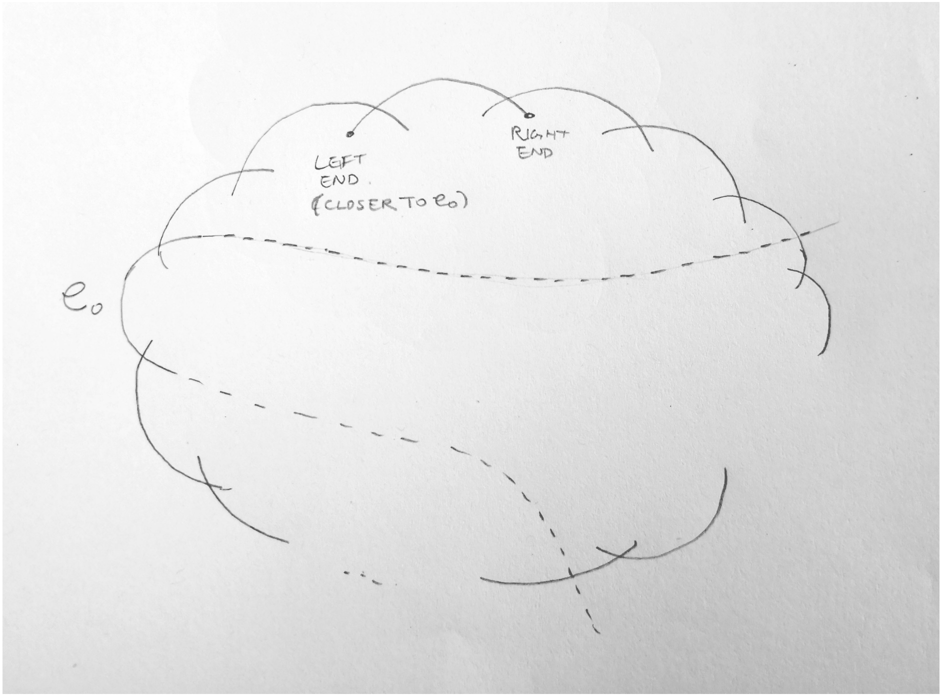


Let the length of C be n. Let e_0 be any string with maximum exit distance on either end, over all strings. Say the maximum exit distance is k. We will label all strings along C as e_0, e_1, e_2, ... until e_{n-1}, with one exit of γ_{e_0} at e_{n-k} and the other at e_i, where $i < n-k$. Let C_1 be the arc $\{e_j : 0 < j < i\}$, C_2 be the arc $\{e_j : i \leq j < n-k\}$, and C_3 be the arc $\{e_j : n-k < j \leq n-1\}$

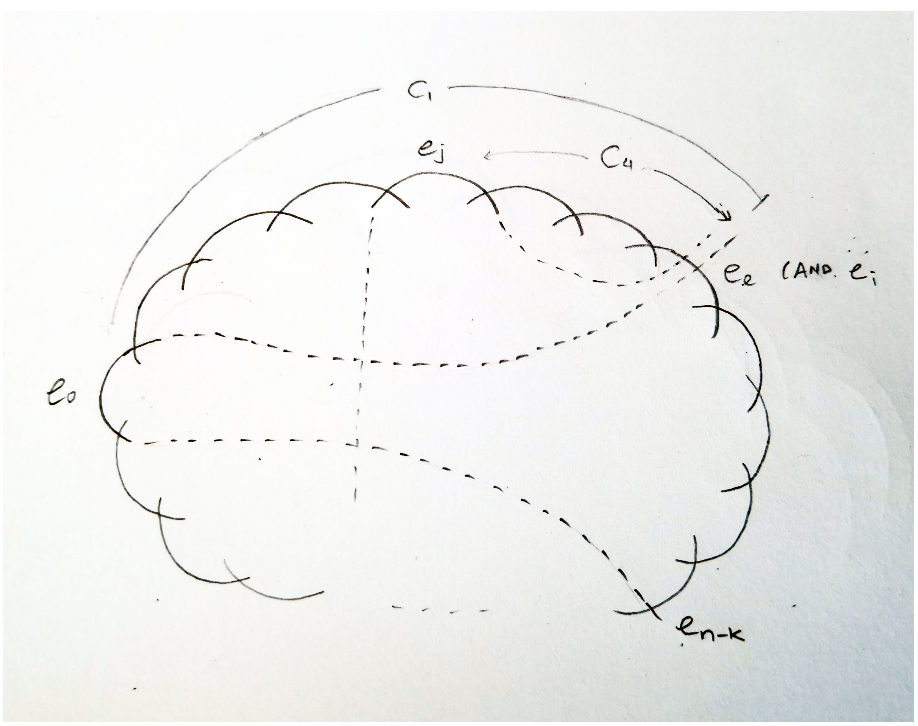


Claim 1: Every string e_j on C_1 has one end exiting on C_1.

Proof: Suppose, for a contradiction, that e_j has both ends exiting outside C_1. If both of its ends exit at C_3 or one end on C_2 and the other on C_3, then we immediately get more than 2 crossings. Otherwise, both ends of e_j exit in C_2, but this implies e_j has an exit distance greater than k, the maximum exit distance, a contradiction. Therefore, e_j has one end exiting on C_1.



Now consider e_j on C_1 with the greatest index and its left end not in C_1. We know there is such an e_j, since if the left end of e_1 exits on C_1, then so does its right end to avoid self-intersection of the pseudocircle. However, both ends of e_1 exiting on C_1 gives us an exit distance larger than maximum. Therefore, such an e_j exists. Let the right end of e_j exit at e_l, and let C_4 be $\{e_m : j < m \leq l\}$



Claim 2: Every string e_m on C_4 has one end exiting on C_4.

Proof: Any string e_m without its left end exiting on C_4 must have its left end exit on $C_1 \setminus C_4$, since its left end exiting from outside C_1 would contradict our assumption of e_j having the highest index or being the rightmost string with its left exit outside C_1. This immediately gives us two intersections of γ_{e_m} and γ_{e_j}, so we must have the right end of γ_{e_m} exiting in C_4 to not intersect γ_{e_j} again.

Furthermore, we must have some e_m without its left end in C_4, since if e_{j+1} has its left end in C_4 it also has its right end in C_4 to avoid self-intersection, and we contradict maximality of k. Therefore, we can keep repeating the previous step to take the rightmost string in the arc with its left exit outside the arc, always maintaining some exit within the arc. At each step, the arc decreases in length. At some step, a string will be forced to exit on its neighbour to violate PS3 as in the lemma, since the arc needs length at least three to prevent this. Thus it is impossible for all strings to be contained entirely within C.

Corollary: We cannot extend all strings outside the cycle when we have fish or shrub cycles either.

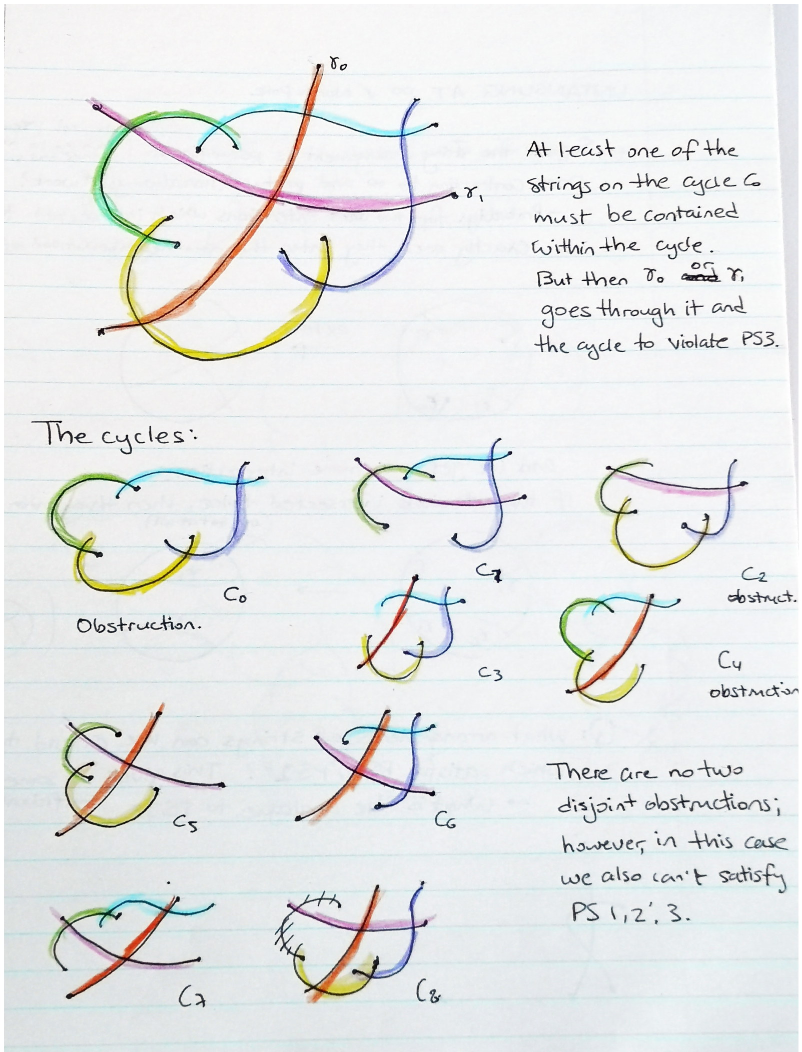
Proof: In the case of fish, this requires us to choose a string that achieves the maximum exit distance among all strings containing the rainbow vertex before its exit. In the case of shrubs, we choose a string achieving the maximum exit distance among all strings containing both adjacent rainbow vertices before its exit. We can guarantee such strings exist by looking at the strings beside the obstruction, which necessarily exit after the obstruction. Then we can find our same extremal extension arc, and assume all strings begin from the interior of the cycle. We cannot make this same observation about croissants, because we would need to guarantee some string contains both vertices before it exits on one side. Indeed, from our example in the previous section and generalization, we can determine that all such cycles with at least 2 strings on either side of the obstructions can be extended without any contained inside the cycle entirely.

We can now answer Q2: When we have two disjoint cycles of types clouds, fish, or shrubs, then we will not be able to satisfy PS2 but can satisfy PS2'. Thus the two sets of conditions are not equivalent.

4 Answering Q3

From our result in answering question 2, we considered whether an arrangement of strings is pseudocircular if and only if it does not contain two disjoint contained cycles that are clouds, shrubs, or fish.

However, this turns out to be an insufficient characterisation, since we can make use of the previous theorem with one contained pseudocircle and PS3 to construct arrangements of strings that are not pseudocircular but also do not admit two disjoint obstruction cycles.



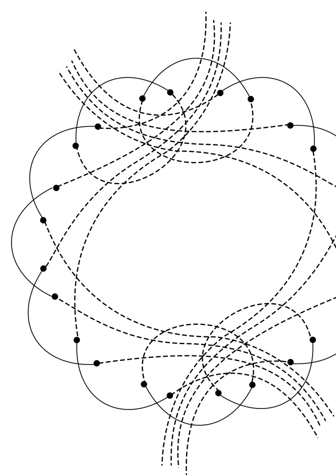
In answering Q2, we showed that we cannot have every string exiting a cloud cycle. Now we will try to determine the minimum number of strings that must be contained in a cloud cycle, in order to characterize the number of strings we need to weave through any cycle in order to prevent a pseudocircular arrangement from PS3. Let $m(C)$ denote the minimum number of pseudocircles contained in the interior of cycle C in a pseudocircular arrangement.

Lemma: When we have an arrangement on an arc of C so that every string has at least one end exiting on the arc and the right ends exit to the right, and the left ends to the left, then at least two of the strings an the arc must be extended to pseudocircles which are entirely contained within the cycle.

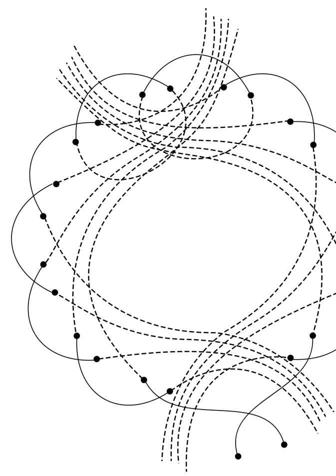
Proof: We will once again take the string with the longest exit distance, which has at least one pseudocircle contained in the cycle from some string e in C. Then we can continue to take smaller arcs. At each step we can find at least one string with its left exit not in the arc, since while the first string from the left may not exit the cycle, the second has its left exit on its left side, and it cannot exit through its neighbour by PS3. Note that the pseudocircle without exits is always in the progression of arcs, so if at any point the pseudocircle we have fixed is not in the arc then we are done. Otherwise we eventually have an arc consisting of two strings, but they must both be pseudocircles since they can only exit on each other, but they cannot exit on their neighbours.

Theorem: Let C be a cloud cycle with length at least 4. Then $m(C) = 4$.

Proof: Using the previous lemma and any string with furthest exit, we can get two pseudo circles contained in the cycle on one side of some string extension. Then we can choose the string e with the furthest exit which also contains at least two strings between the string and the exit string which extend to pseudocircles contained in the cycle, since we have existence. Then we can use this extremal to get two more pseudocircles extending from strings between e and the other exit. We can show that $m(C) \leq 4$ with the following extension:



Theorem: Let C be a fish or shrub cycle with length at least 4. Then $m(C) = 2$.



Remark: All pseudolinear arrangements of strings are pseudocircular. We can contract all infinite ends of these lines to a point, and perturb it slightly to obtain two intersections for every pseudocircle pair while keeping all intersections distinct.

References

[1] Alan Arroyo, Julien Bensmail, and R. Bruce Richter. Extending drawings of graphs to arrangements of pseudolines, 2018.